Saffman, P. G., "Dispersion due to Molecular Diffusion and Macroscopic Mixing in Flow through a Network of Capillaries," *J. Fluid Mech.*, 7, 194 (1960).
Scheidegger, A. E., "The Physics of Flow through Porous Media," pp. 112-155, Macmillan, New York (1961).
Snyder, L. J., and W. E. Stewart, "Velocity and Pressure Profiles for Newtonian Creeping Flow in Regular Packed Beds of Spheres," *AIChE J.*, 12, 167 (1966).
Spielman, L. A., and S. L. Goren, "Capture of Small Particles by London Forces from Low-Speed Liquid Flows," *Environ. Sci. Technol.*, 4, 135 (1970).
Wicksell, S. D., "The Corpuscle Problem. A Mathematical Study of a Biometric Problem," *Biometrica*, 17, 84 (1925).
Yao, K.-M., M. T. Habibian, and C. R. O'Melia, "Water and Waste Water Filtration: Concepts and Applications," *Environ. Sci. Technol.*, 5, 1105 (1971).

Manuscript received May 16, 1972; revision received August 10, 1972; paper accepted August 11, 1972.

Part II. Numerical Solution of Steady State Incompressible Newtonian Flow Through Periodically Constricted Tubes

A numerical method for the solution of the problem of steady state, incompressible Newtonian flow through periodically constricted tubes is developed. All terms of the Navier-Stokes equation are retained, including the nonlinear inertia terms.

Sample calculations for a uniform periodically constricted tube, the geometry of which is connected with the modeling of a packed bed of sand are given, including streamlines, axial and radial velocity profiles, pressure profiles, and the dimensionless pressure drop versus Reynolds number relation. The effect of some geometric characteristics of periodically constricted tubes on their friction factor is investigated numerically, and comparison of some existing experimental data with calculated ones is made.

ALKIVIADES C. PAYATAKES
CHI TIEN and
RAFFI M. TURIAN
Department of Chemical Engineering and Materials Science
Syracuse University
Syracuse, New York 13210

SCOPE

The main purpose of this work is the development of a numerical method for the solution of the laminar steady state flow of an incompressible Newtonian fluid through a periodically constricted tube. This problem arises in the modeling of porous media. Petersen (1958) and Houpeurt (1959) have introduced capillaric models involving periodically constricted tubes. In Part I of this paper, the authors developed a statistical model for randomly packed beds of monosized or nearly monosized grains involving unit cells, each one of which resembles a segment of a periodically constricted tube, and they have proposed that the flow through any one of these unit cells can be identified with the flow through a segment of an infinitely long periodically constricted tube formed by the repetition of the unit cell under consideration. The detailed knowledge of the flow through the unit cells is necessary for the calculation of the permeability as well as for the modeling of any process which takes place in the void space of a packed bed, for example, the filtration of suspensions through packed beds encountered in water and waste water renovation and the filtration of feedstocks to catalytic beds in the oil industry. Beyond these specific applications, the solution of the problem will most probably be of use in future work in the modeling of porous media. The formulation of statistical models for anisotropic porous media involving constricted-tube type unit cells does not seem to be beyond reach, and the present work would be directly applicable to such models. Furthermore, application of the present work is not confined only to the case of periodically constricted tubes; the developed algorithm can be used to solve the problem through any tube of nonuniform cross section, such as venturi tubes, partially clogged arterial flows, tubes with step changes of diameter, and so on.

CONCLUSIONS AND SIGNIFICANCE

A numerical method was developed for the solution of the laminar steady state flow of an incompressible Newtonian fluid through both uniform and nonuniform periodically constricted tubes. The (nonlinear) inertia terms of the Navier-Stokes equation are retained. The solution is obtained in terms of the stream function and the nonvanishing component of the vorticity vector. Based on this information, the axial and radial components of the velocity vector as well as the pressure distribution are readily obtainable. In terms of the stream function and vorticity the algorithm is of second order everywhere, except in the immediate vicinity of the wall, where the special discretization procedure results in a scheme which is, in general, of first order. The stability of the algorithm can be assured by proper adjustment of two weighting factors involved in the scheme. Sample calculations are carried

out for the case of a uniform periodically constricted tube the geometry of which is connected with the modeling of a packed bed of sand (See Part I). The streamlines, axial and radial velocity profiles, pressure profiles, and the *dimensionless* pressure drop versus Reynolds number relation are given. The effect of some geometric characteristics of periodically constricted tubes on their friction factor is investigated numerically, and comparison between existing experimental friction factor values and calculated ones is made, resulting in fair agreement.

The most obvious application of this work lies in the field of flow through porous media of the type which can be modeled with constricted-tube-type unit cells or periodically constricted capillaries. The algorithm can, more generally, be used to determine the flow through any tube with nonuniform cross section.

LITERATURE SURVEY

The idea of using periodically constricted tubes to model the void space of porous media was introduced almost simultaneously by Petersen (1958) and Houpeurt (1959) and was utilized to develop a model for the void space of granular porous media in Part I of this paper. Studies on the flow of fluids through periodically constricted tubes are rather limited. Konobeev and Zhavoronkov (1962) conducted experimental investigations of turbulent gas flow through periodically constricted tubes of various wave lengths and amplitudes and established approximate friction factor expressions for limiting cases. More recently, Hsu (1968) studied the turbulent flow in wavy tubes. Miller et al., (1967) investigated the effect of nonuniformity of the cross section on the flow through capillaries. Batra (1969) conducted a series of experimental investigations of laminar flow at intermediate and high Reynolds numbers through flexible wavy tubes and rigid wavy channels and reported experimental friction factor-Reynolds number relationships. Several sets of Batra's data will be compared to the numerical results of the present work.

A significant number of numerical solutions of the Navier-Stokes equation have been published in recent years and they are too numerous to be cited individually. For work of the kind presently considered, it is common to express the Navier-Stokes equation in terms of the stream function as the dependent variable. Bye (1965) developed a line iteration scheme for the solution of the complete two-dimensional Navier-Stokes equation expressed in terms of the stream function and used it to solve a flow problem in a basin of rectangular cross section. Christiansen et al. (1972) reported a numerical solution of the Navier-Stokes equation expressed in terms of the stream function for the case of laminar tube flow through an abrupt contraction. The discretization of the complete Navier-Stokes equation written in terms of the stream function leads to a cumbersome numerical scheme. The complications are greatly increased when the domain of interest is not rectangular (or cannot be transformed into a rectangular one by an appropriate change of variables) but has curved boundaries. For this reason the approach introduced by Thom (1929) applicable to two-dimensional flows, has been adopted by a growing number of investigators. This approach, known as the stream function-vorticity method, consists of the replacement of the 4th-order partial differential equation in terms of the stream function with a system of two partial differential equations, each of second order, with the stream function and the nonvanishing component of the vorticity as the dependent variables. One of these equations is linear and the other, containing the inertia terms, nonlinear. This approach has been used to obtain numerically the solution to both transient (Fromm and Harlow, 1963; Rimon and Cheng, 1969; Son and Hanratty, 1969) and steady state (Christiansen et al., 1972; Greenspan, 1969; Hamielec et al., 1967; Hamielec, Johnson, et al., 1967; Hamielec and Raal, 1969; LeClair and Hamielec, 1968, 1970; Masliyah and Epstein, 1971; Mehta and Lavan, 1969; Runchal and Wolfstein, 1969; Vrentas et al., 1966; Wang and Longwell, 1964) flow problems. A variation of the stream function-vorticity method has been adopted in the present work.

GEOMETRY

Two types of periodically constricted tubes will be considered. The first can be thought to be generated by connecting in series an infinite number of identical tube segments and will be called uniform periodically constricted type (UPCT). For the purpose of numerical calculation, only a simple segment preceded and followed by an identical half segment is needed, Figure 1a. Each of the segments of a UPCT is assumed to have axial symmetry, and also plane symmetry, with the plane of symmetry being the one containing its narrowest cross section. The wall of a UPCT segment is obtained by the revolution of a curve segment which is assumed to be piecewise smooth. In Part I of this series it has been argued that the flow through each unit cell of the model can be identified with the flow through a UPCT formed by the repetition of the unit cell under consideration. Such UPCT's have segments with walls which are parabolae of revolution. The second type, called nonuniform periodically constricted tube (NPCT), is generated from a UPCT if a single segment of the latter is replaced with another one, of equal length, but otherwise different geometry. NPCT's with segment walls which are parabolae of revolution arise in connection with a modified version of the model of Part I of this series (Payatakes, 1972). Since the flow through a NPCT far away from the region of nonuniformity tends to that through the corresponding UPCT, one needs to concentrate on the region of nonuniformity, Figure 1b.

MATHEMATICAL FORMULATION AND METHOD OF SOLUTION

Let (z^*, r^*, θ) be the dimensionless cylindrical polar coordinates using as characteristic length the length of a segment of the periodically constricted tube. Consider the region OABCDEFGHIO on the plane $(z^*, r^*, 0)$, as shown in Figure 2, which constitutes the domain of interest. Let O coincide with the origin of the coordinates and OIHGF lie on the z^*-axis. The curved segment ABCDE represents the section of the wall of the constricted tube with the plane $(z^*, r^*, 0)$. The radial distance from the wall to the axis is a function of z^* and is denoted by $r_w^*(z^*)$. The mathematical formulation and the numerical algorithm are valid for any form of $r_w^*(z^*)$ which is piecewise smooth. For the particular case of the periodically constricted tubes used for the modeling of the void space of porous media (Payatakes, Tien, and Turian, 1972; Payatakes, 1972) $r_w^*(z^*)$ takes the form

$$r_w^*(z^*) = \begin{cases} & \text{for} \quad 0 \le z^* \le \dfrac{1}{2} \\ r_3^* + 4(r_2^* - r_3^*)(1 - z^*)^2 & \\ & \text{for} \quad \dfrac{1}{2} \le z^* \le \dfrac{3}{2} \\ r_1^* + 4(r_2^* - r_1^*)(2 - z^*)^2 & \\ & \text{for} \quad \dfrac{3}{2} \le z^* \le 2 \end{cases} \tag{1}$$

Equation (1) is based on the assumption that the curved sections AB, BCD, and DE belong to three parabolae with apices coinciding with A, C, and E, respectively. Through rotation of the curve ABCDE around the z^*-axis, a segment of a periodically constricted tube is obtained, which consists of a unit cell [that is, the part corresponding to IBCDGHI] preceded and followed by a half unit cell [that is, the parts corresponding to OABIO and GDEFG, respectively]. Depending upon whether the mid constriction is equal to the end constrictions or not [$r_1^* = r_3^*$ or $r_1^* \neq r_3^*$], the configuration is denoted as UPCT or NPCT, respectively.

Due to the axisymmetry of the configuration, the flow is two dimensional. Let ψ be the stream function. The axial velocity component v_z and the radial velocity component v_r are given by

$$v_z = -\frac{1}{r}\frac{\partial\psi}{\partial r} \quad \text{and} \quad v_r = \frac{1}{r}\frac{\partial\psi}{\partial r} \tag{2}$$

where (z, r) are the dimensional counterparts of (z^*, r^*). Let the dimensionless velocity components, stream function, and pressure be defined as

$$v_z^* = \frac{v_z}{v_0}, \; v_r^* = \frac{v_r}{v_0}, \; \psi^* = \frac{\psi}{\tau^2 v_0}, \; P^* = \frac{P}{\rho v_0^2} = \frac{p - \rho g z}{\rho v_0^2} \tag{3}$$

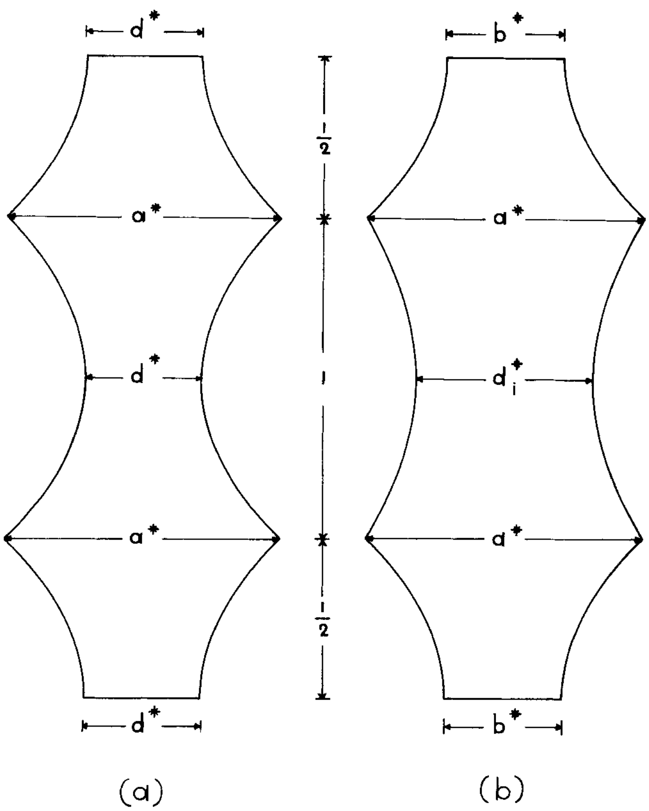

Fig. 1. (a). Section of a uniform periodically constricted tube; (b). Section of a nonuniform periodically constricted tube.

$$\tau = \frac{z}{z^*} = \frac{r}{r^*} \tag{4}$$

$$v_0 = \langle v_z(0, r) \rangle = \frac{q}{\pi(r_1^* \tau)^2} \tag{5}$$

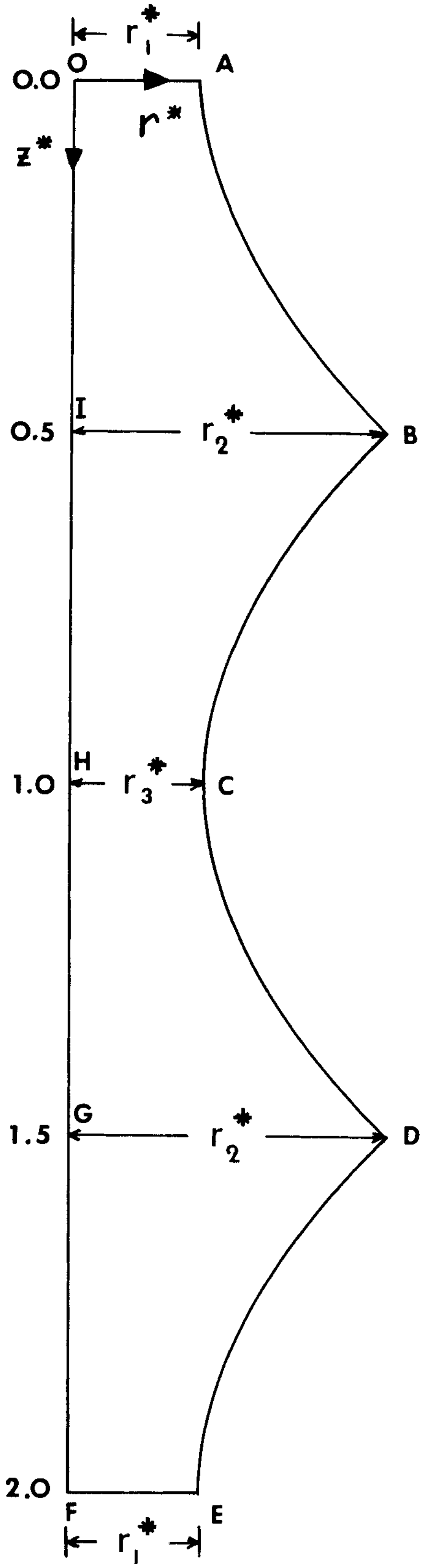

Fig. 2. Extended dimensionless unit cell. Domain of interest.

relations:

$$v_z^* = -\frac{1}{r^*}\frac{\partial\psi^*}{\partial r^*}, \quad v_r^* = \frac{1}{r^*}\frac{\partial\psi^*}{\partial z^*} \tag{6}$$

The system of partial differential equations that govern the steady state axisymmetric flow of a Newtonian incompressible fluid in terms of the stream function ψ^* and the quantity $\omega^* = r^* w_\theta^*$ (where w_θ^* is the only nonvanishing component of the dimensionless vorticity vector $\underline{w}^* = \nabla^* \times \underline{v}^*$), is

$$E^{*2}\psi^* = \omega^* \tag{7}$$

$$E^{*2}\omega^* = -N_{Re}\left[\frac{1}{r^*}\frac{\partial(\psi^*,\omega^*)}{\partial(r^*,z^*)} + \frac{2}{r^{*2}}\frac{\partial\psi^*}{\partial z^*}\omega^*\right] \tag{8}$$

where

$$E^{*2} \equiv \frac{\partial^2}{\partial z^{*2}} - \frac{1}{r^*}\frac{\partial}{\partial r^*} + \frac{\partial^2}{\partial r^{*2}} \tag{9}$$

and

$$N_{Re} = \frac{r v_0}{\nu} \tag{10}$$

The above system is of the elliptic type and its solution requires prescription of boundary conditions everywhere along the boundary OABCDEFGHIO of the domain of interest. The boundary conditions are:

Along Axis OF

$$\psi^*(z^*, 0) = \frac{1}{2}r_1^{*2} \tag{11}$$

$$\omega^*(z^*, 0) = 0 \tag{12}$$

Along Wall, ABCDE

$$\psi^*[z^*, r_w^*(z^*)] = 0 \tag{13}$$

$$\omega^*[z^*, r_w^*(z^*)] = \left(\frac{\partial^2}{\partial z^{*2}} + \frac{\partial^2}{\partial r^{*2}}\right)\psi^*[z^*, r_w^*(z^*)] \tag{14}$$

Along Entrance, OA, and Outlet, FE

The boundary conditions at the entrance and outlet are not known. Given the geometry of the tube and the value of the Reynolds number these conditions can, in principle, be determined uniquely, but they are not known from the outset. Actually, these boundary conditions are part of the solution rather than part of the statement of the problem. Nevertheless, before the numerical solution can be undertaken, $\psi^*(0, r^*)$, $\psi^*(2, r^*)$, $\omega^*(0, r^*)$ and $\omega^*(2, r^*)$ must be specified. Let

$$\psi^*(0, r^*) = f_1^*(r^*; r_1^*) \tag{15}$$

$$\psi^*(2, r^*) = f_2^*(r^*; r_1^*) \tag{16}$$

$$\omega^*(0, r^*) = f_3^*(r^*; r_1^*) \tag{17}$$

$$\omega^*(2, r^*) = f_4^*(r^*; r_1^*) \tag{18}$$

Since f_1^*, f_2^*, f_3^*, and f_4^* are not given in advance, the solution of the system is obtained by an iterative procedure.

For the case of the flow through a UPCT, if one assumes that a steady state periodic solution exists, it follows that the velocity field in any segment is identical to that in any other segment. Accordingly, one has

$$\psi^*(0, r^*) = \psi^*(1, r^*) = \psi^*(2, r^*) = f_1^*(r^*; r_1^*) = f_2^*(r^*; r_1^*) \tag{19}$$

and

$$= f_3^*(r^*; r_1^*) = f_4^*(r^*; r_1^*) \tag{20}$$

One can, therefore, adopt the procedure of using the profiles $\psi^*(1, r^*)$ and $\omega^*(1, r^*)$ obtained from the $(i-1)$st iteration as approximations of f_1^* and f_3^* for the ith iteration. To initiate this scheme, f_1^* and f_3^* are taken to be

$$f_1^{*(0)}(r^*; r_1^*) = \frac{1}{2}r_1^{*2}\left[1 - \left(\frac{r^*}{r_1^*}\right)^2\right]^2 \tag{21}$$

$$f_3^{*(0)}(r^*; r_1^*) = 4\left(\frac{r^*}{r_1^*}\right)^2 \tag{22}$$

which are the expressions for the fully developed flow in a smooth tube of radius r_1^* (Poiseuille flow). An alternate approach would be to set $f_1^{*(0)}$ and $f_3^{*(0)}$ equal to the profiles of $\psi^*(1, r^*)$ and $\omega^*(1, r^*)$ from the solution for a different Reynolds number, if such a solution is available.*

This iteration is continued until the conditions of Equations (19) and (20) are satisfied. This procedure converges rapidly; in most cases, only two iterations are enough for the determination of a solution that satisfies the requirement of periodicity with an accuracy which is satisfactory for all practical purposes. A third iteration renders results which differ only negligibly from those of the second iteration. Further iterations do not produce changes of any consequence.

The case of flow through a NPCT is somewhat more complicated. Such a tube is obtained if one segment of a UPCT is replaced by a segment of equal length but otherwise different geometry. Since far away from the region of nonuniformity the flow solution tends to the one through the corresponding UPCT, for the same value of N_{Re}, a rigorous approach would require the solution of the problem in a domain which is obtained from that depicted in Figure 2 by addition of at least one more segment, both upstream and downstream. These extensions should be long enough so that the flow solution in the first two consecutive segments at both ends is periodic. This procedure requires rather large memory and computing time and is, therefore, impractical. Instead, the following approach is preferred. The problem of flow through the corresponding UPCT is solved first, as described above. Let $f_{U1}^*(r^*; r_1^*)$ and $f_{U3}^*(r^*; r_1^*)$ be the profiles of $\psi^*(1, r^*)$ and $\omega^*(1, r^*)$, respectively, obtained from this solution. The solution of the system of Equations (7), (8), (11) to (18) for the NPCT is obtained after setting

$$f_1^*(r^*; r_1^*) = f_2^*(r^*; r_1^*) = f_{U1}^*(r^*; r_1^*) \tag{23}$$

$$f_3^*(r^*; r_1^*) = f_4^*(r^*; r_1^*) = f_{U3}^*(r^*; r_1^*) \tag{24}$$

The solution thus obtained is only approximate (even after discounting discretization and round-off errors), due to the approximation of Equations (23) and (24). However, this approximation is a satisfactory one, especially in the region $1/2 \leq z^* \leq 3/2$, since the sections for $0 \leq z^* \leq 1/2$ and $3/2 \leq z^* \leq 2$ serve to partially iron out the discrepancies introduced by the approximate nature of the entrance and outlet boundary conditions. This was verified for the special case: $r_1^* = r_2^* = 0.6$, $r_3^* = 0.3$, $N_{Re} = 1$, using $\kappa^* = 1/24$. The results, which were obtained using the above mentioned approximation and those obtained by adding one full segment at each end, were compared. The maximum difference for the values of ψ^* in the seg-

* This approach is particularly useful when solutions are required for several values of the Reynolds number.

ment containing the mid constriction, was found to be less than 0.6% of $\psi^*(z^*, 0)$.

ALGORITHM FOR THE NUMERICAL SOLUTION OF THE SYSTEM OF EQUATIONS (7), (8), (11) TO (18)

The algorithm developed in this work parallels in certain aspects the one developed by Greenspan (1969) for the solution of the flow in a channel of rectangular cross section with a single step, with the important difference that whereas Greenspan's algorithm is of the first order, the present algorithm is of the second order everywhere in the domain, except on the network nodes in the immediate vicinity of the wall where a special discretization scheme had to be devised to accommodate the irregular shape of the boundary. The algorithm, due to its length, is given in a Supplement.* The developed algorithm displays good stability even for high values of the Reynolds number. This stability is due to the fact that the systems of linear equations which are solved iteratively during the solution of the problem have been constructed so that they have matrices of non-negative type and with diagonal dominance irrespective of the value of the Reynolds number, properties which although do not stop the propagation of error, they do not allow its growth either. Instability may arise in the outer iteration (for definition and details see Supplement) for high values of the Reynolds number. In such cases, convergence can be achieved by adjustment of the two weighting factors involved in the scheme, at least this was the case in all iterations encountered in this work. The algorithm for the determination of v_z^*, v_r^*, and P^* is also given in the same Supplement.

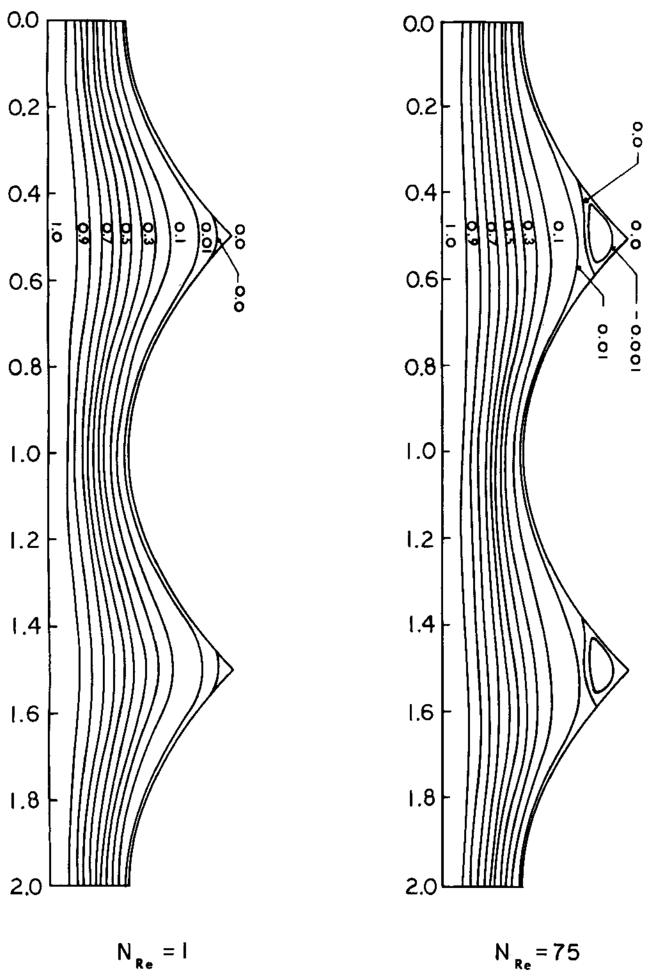

Fig. 3. Streamlines of flow in a UPCT with $a^* = 0.860$ and $d^* = 0.364$.

* Supplement has been deposited as Document No. 01996 with the National Auxiliary Publications Service (NAPS), c/o Microfiche Publications, 305 East 46 Street, New York 10017 and may be obtained for $2.00 for microfiche and $5.00 for photocopies.

SAMPLE CALCULATIONS

Calculated results are presented below for a UPCT each segment of which is identical to the dimensionless unit cell corresponding to a packed bed of sand 20 × 30, according to the model postulated by Payatakes, Tien and Turian (1972)

Cell geometry
$$r_1^* = r_3^* = \frac{1}{2}d^* = 0.182$$
$$r_2^* = \frac{1}{2}a^* = 0.430$$
walls: parabolae of revolution [Equation (1)]

The flow was determined for $N_{Re} = 1$, 10, 30, 50, 75. It is convenient to define a normalized stream function ψ^+ and a normalized pressure P^+ as follows:

$$\psi^+ \equiv \frac{\psi^*}{\psi^*(z^*, 0)} = \frac{\psi^*}{\frac{1}{2}r_1^{*2}} \tag{25}$$

$$P^+ \equiv \frac{P^*(z^*, r^*) - P^*\left(\frac{3}{2}, 0\right)}{P^*\left(\frac{1}{2}, 0\right) - P^*\left(\frac{3}{2}, 0\right)} = 1 - \frac{P^*}{\Delta P^*} \tag{26}$$

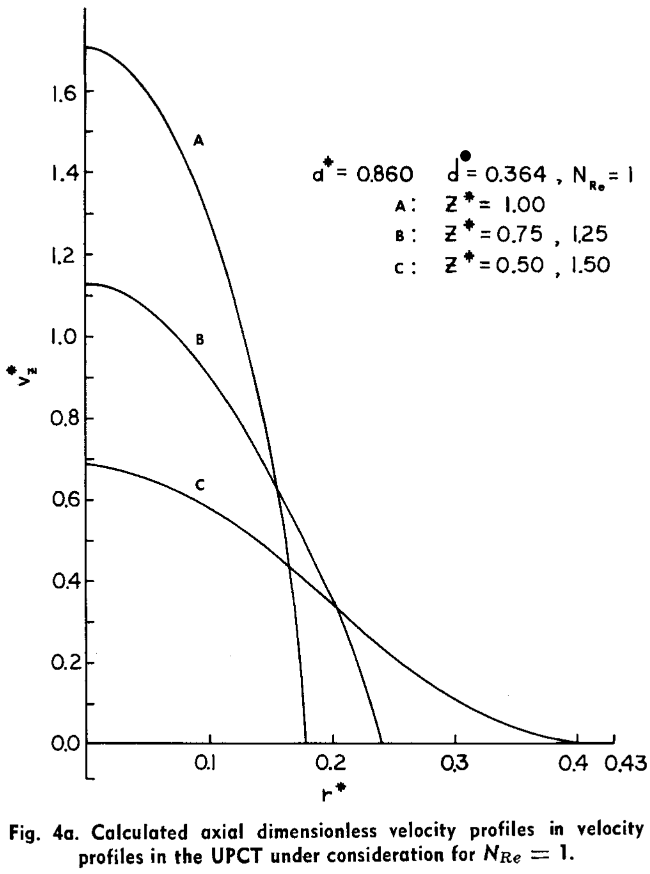

Fig. 4a. Calculated axial dimensionless velocity profiles in velocity profiles in the UPCT under consideration for $N_{Re} = 1$.

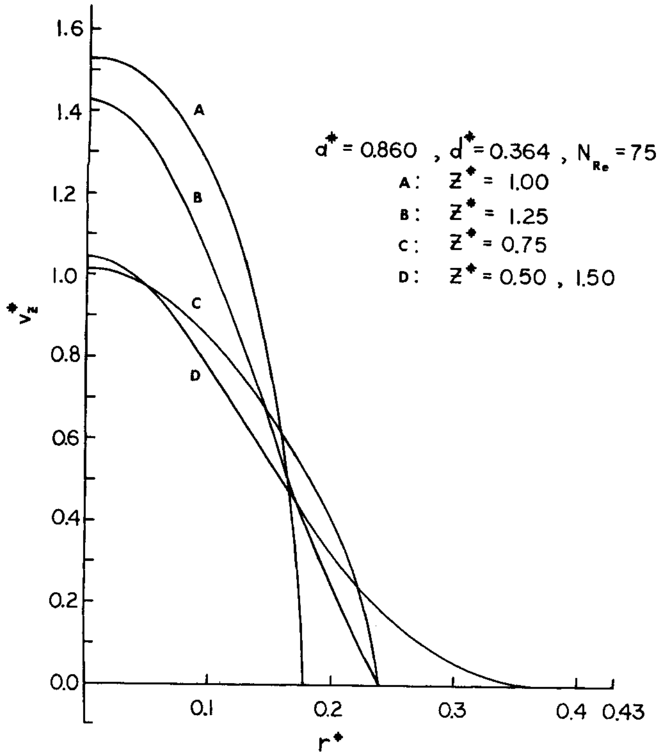

Fig. 4b. Calculated axial dimensionless velocity profiles in the UPCT under consideration for $N_{Re} = 75$.

where we have set

$$-\Delta P^* = P^*\left(\frac{1}{2},0\right) - P^*\left(\frac{3}{2},0\right), \quad P^*\left(\frac{1}{2},0\right) \equiv 0 \tag{27}$$

Calculated streamlines for given values of ψ^+, axial velocity profiles, radial velocity profiles, and normalized pressure profiles for selected values of N_{Re} are given in Figures 3 to 6, respectively. The calculated relation between $(-\Delta P^*)$ and N_{Re} is given in Figure 7. It can be seen that for small N_{Re} this relation is given by

$$\Delta P^* = \frac{\Delta P_1^*}{N_{Re}} \qquad N_{Re} < 5 \tag{28}$$

where ΔP_1^* is the dimensionless pressure drop at $N_{Re} = 1$. For the particular geometry under consideration $\Delta P_1^* = -101.1$, $(\kappa^* = 1/36)$. Equation (28) holds only approximately for higher values of N_{Re}. The deviation of the value of $(-\Delta P^*)$ given by Equation (28) from the one computed numerically for $N_{Re} = 10, 30, 50, 75$ is -0.3%, -2.4%, -5.43%, and -8.95%, respectively. The computations were carried out with an IBM 370/155 digital computer. The values of the overrelaxation factors, weighing factors and tolerances used, as well as the required CPU times for $\kappa^* = 1/36$, are given in Table 1.*

A necessary test of any numerical algorithm concerns its convergence. A simple empirical test can be made by comparing the results of a calculation carried out with a certain increment size to those obtained from a second calculation carried out with a substantially smaller increment size. If the results are not significantly different, the scheme is considered to be convergent; otherwise, the calculation should be repeated with the use of decreasing steps until no further significant changes are encountered. This approach, however, requires extensive computations.

An alternate method can be used, whenever the numerical scheme under consideration can be applied to a problem of which an exact solution is available. This method provides both a test of convergence and the proper step size for the required accuracy, but it suffers from the disadvantage that the test pertains to a problem which is similar but not the same with the problem at hand. In the present case the Poiseuille flow in a cylindrical tube offers

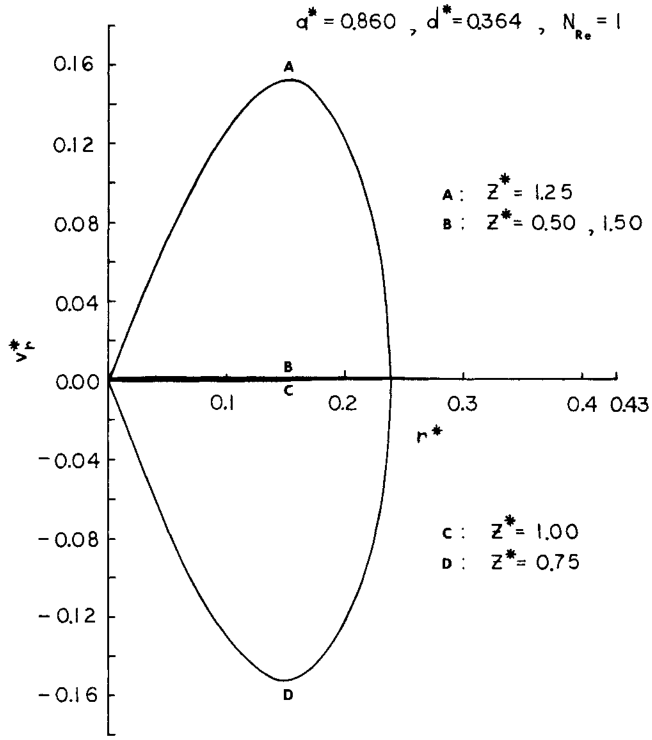

Fig. 5a. Calculated radial dimensionless velocity profiles in the UPCT under consideration for $N_{Re} = 1$.

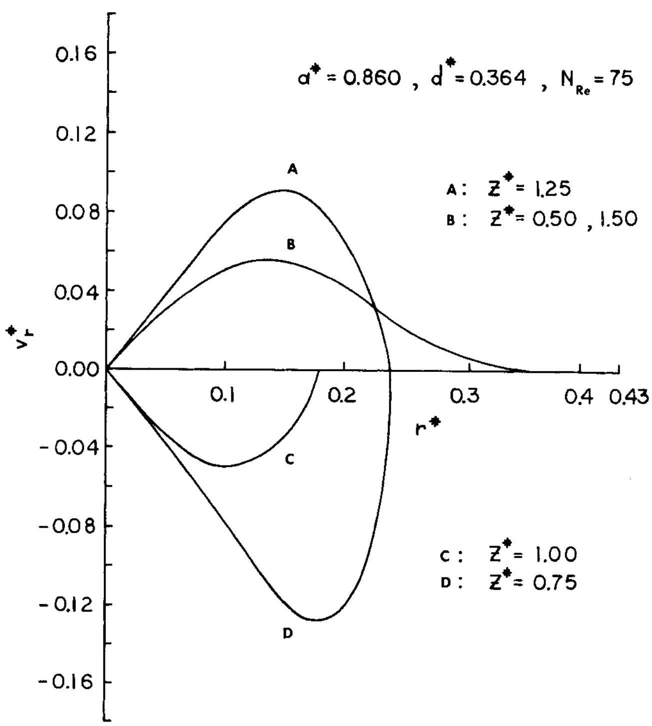

Fig. 5b. Calculated radial dimensionless velocity profiles in the UPCT under consideration for $N_{Re} = 75$.

* See footnote on page 71.

itself as testing ground for the algorithm developed in the preceding sections. The flow through a smooth cylindrical tube with diameter $a^* = d^* = 2.225$ was solved numerically with $\kappa^* = 1/12$. This choice of geometry and step is such that there are 15 nodes on each row and the last node is irregular. The maximum difference of the calculated stream function values from the exact ones is $+2.8\%$ of $\psi^*(z^*, 0)$. The maximum difference of the calculated values for the axial velocity from the exact ones appears on the axis where this difference is -6.2% of the exact value. This discrepancy should be expected given that velocities are calculated based on values of the stream function which are correct only to the order $0(\kappa^{*2})$. The calculated pressure drop value differs from the exact value by -3.34%. This agreement is remarkably good taking into consideration that velocity values, which are correct only to the order $0(\kappa^*)$, and their first and second derivatives, obtained using central difference expressions, are used in the calculation of ΔP^*.† It seems that this should be attributed to the fact that the scheme used for the

TABLE 1. INFORMATION RELATING TO THE NUMERICAL SOLUTION OF THE FLOW THROUGH A UPCT WITH $a^* = 0.860$ AND $d^* = 0.364$ FOR $\kappa^* = 1/36$

N_{Re}	ζ_ψ	ζ_ω	ρ_ψ	ρ_ω	e_ψ	e_ω	CPU Time† min.
1	1.5	1.5	0.3	0.5	10^{-8}	10^{-5}	2.3
10	1.5	1.5	0.3	0.5	10^{-8}	10^{-5}	2.0
30	1.5	1.5	0.3	0.5	10^{-8}	10^{-5}	2.0
50	1.5	1.5	0.3	0.70	10^{-8}	10^{-5}	3.3
75	1.5	1.5	0.3	0.80	10^{-8}	10^{-5}	6.3

† An object deck was used, and the computations were carried with double precision.

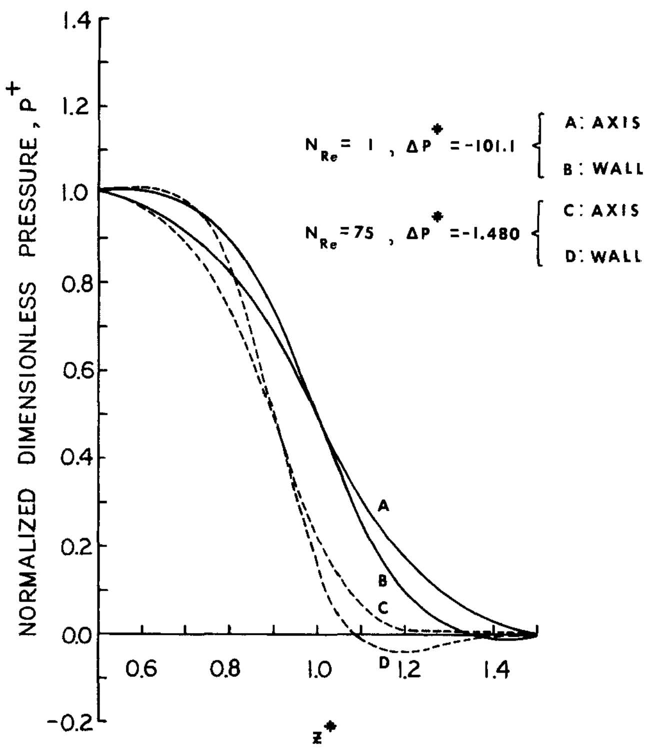

Fig. 6. Profiles of normalized dimensionless pressure on the axis and on the wall.

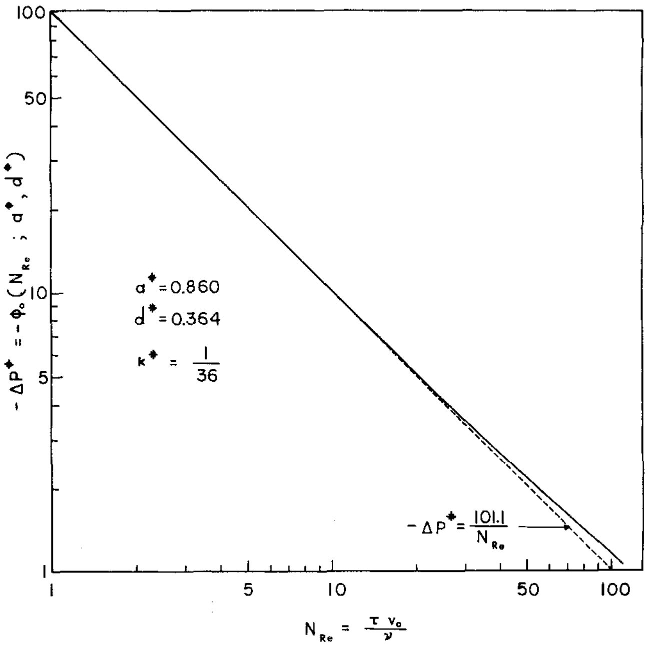

Fig. 7. Friction factor as a function of the Reynolds number for the UPCT under consideration.

numerical integration of the pressure [Simpson's rule] is of the order $0(\kappa^{*5})$, when the values of $\partial P^*/\partial r^*$ and $\partial P^*/\partial z^*$ are exact.

From the above considerations, the following rule of thumb could be drawn: comparable accuracy can be expected when the step is so chosen that there are at least 15 nodes on the row corresponding to the smaller diameter of the periodically constricted tube. Unfortunately, in many cases in this work, adherence to this rule would lead to excessively long computations, due to the large r_2^*/r_1^* value. In such cases a larger step was chosen at the expense of accuracy.

EFFECT OF VARIOUS GEOMETRIC FACTORS ON THE PRESSURE DROP AND COMPARISON WITH EXPERIMENTAL RESULTS

In order to assess the effect on pressure drop due to the various types of constrictions, consider a UPCT the wall of which is formed by revolution of the curve defined by

$$r_w^* = r_1^* + 4(r_2^* - r_1^*)(3 - 4z^*)z^{*2} \quad \text{for } 0 \le z^* \le \frac{1}{2}$$
$$= r_2^* - (r_2^* - r_1^*)(5 - 4z^*)(2z^* - 1)^2 \quad \text{for } \frac{1}{2} \le z^* \le 1 \qquad (29)$$

Equation (29) generates a smooth curve that passes through the points $(0, r_1^*)$, $(1/2, r_2^*)$, $(1, r_1^*)$, [on the plane $(z^*, r^*, 0)$], and also has first derivatives that vanish at these points. Let d_v be the volumetric diameter, defined as the diameter of a cylindrical smooth tube which has the same volume per unit length as the UPCT under consideration. Let

$$\Delta r^* = r_2^* - r_1^* \qquad (30)$$

It can be demonstrated that when Equation (29) holds

$$r_1^* = r_3^* = \frac{1}{2}\left[\sqrt{d_v^{*2} - \frac{17}{35}(\Delta r^*)^2} - \Delta r^*\right] \qquad (31)$$

* See footnote on page 71.

$$r_2^* = \frac{1}{2}\left[\sqrt{d_v^{*2} - \frac{17}{35}(\Delta r^*)^2} + \Delta r^*\right] \quad (32)$$

where

$$d_v^* = \frac{d_v}{\tau} \quad (33)$$

Let

$$v_m = \frac{4q}{\pi d_v^2} \quad (34)$$

and

$$(N_{Re})_v = \frac{v_m d_v}{\nu} \quad (35)$$

Then

$$(N_{Re})_v = \frac{4r_1^{*2}}{d_v^*} N_{Re} \quad (36)$$

With the definition

$$f_v = -\frac{(\Delta P) d_v}{2\tau \rho v_m^2} \quad (37)$$

for the friction factor of the UPCT, it can be demonstrated that

$$f_v = -\frac{d_v^{*5}}{32 r_1^{*4}} \Delta P^* \quad (38)$$

An example of the effect of variations of the value of Δr^* with $d_v^* =$ const. on f_v is given in Figure 8. In Supplement,† the streamlines for two UPCT's with common d_v^* but different Δr^* are depicted. An example of the effect of variations of d_v^* with $\Delta r^* =$ const. on f_v is given in Figure 9. In Figure 13 of the Supplement, the streamlines for two UPCT's with the same Δr^* but different d_v^* are depicted.

Experimental information concerning laminar flow in periodically constricted tubes is rather scarce. In Figures 10 and 11 experimental values of the friction factors through periodically constricted tubes obtained by Batra (1969) are compared with calculated values. This comparison is not entirely rigorous since the geometry of the walls of the tubes used by Batra was neither completely periodic nor exactly defined due to the presence of substantial irregularities. The calculated values have been obtained based on UPCT's with d_v^* and Δr^* which are equal to the corresponding average experimental values and wall which is described by Equation (29). The calculated and experimental values of f_v are in fair agreement. It is, however, pertinent to point out that a conclusion reached by Batra (1969) based on experimental evidence and reiterated by Dullien and Batra (1970) namely that "The value of the friction factor increases with decreasing wave length to diameter ratio (< 0.5) by as much as 120% of the uniform tube value (wave amplitude to diameter ratios were in the range 0.13 to 0.25)" seems to be incorrect. In the notation of the present work "wave length to diameter ratio" is equal to $\tau/d_v = 1/d_v^*$ and, as it can be seen in Figure 9, a decrease in $1/d_v^*$, that is, an increase in

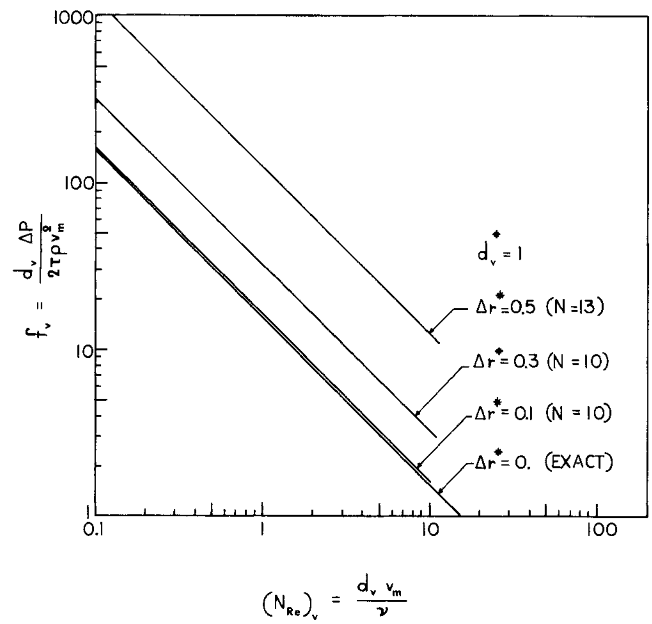

Fig. 8. Effect of the variation of the amplitude on the dimensionless pressure drop for constant volumetric diameter.

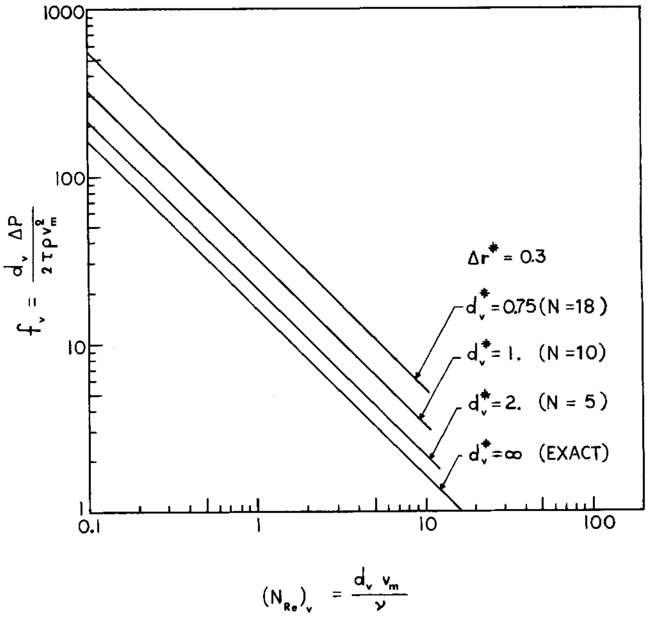

Fig. 9. Effect of the variation of the volumetric diameter on the friction factor for constant amplitude.

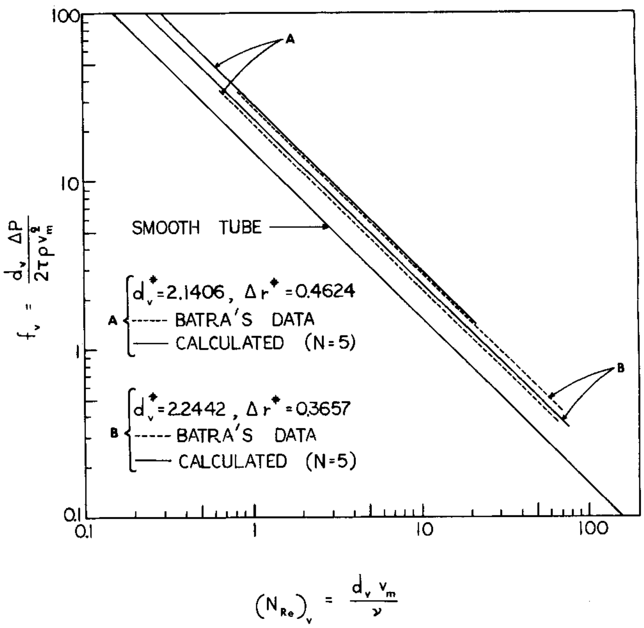

Fig. 10. Comparison of experimental and calculated values of friction factors of UPCT's.

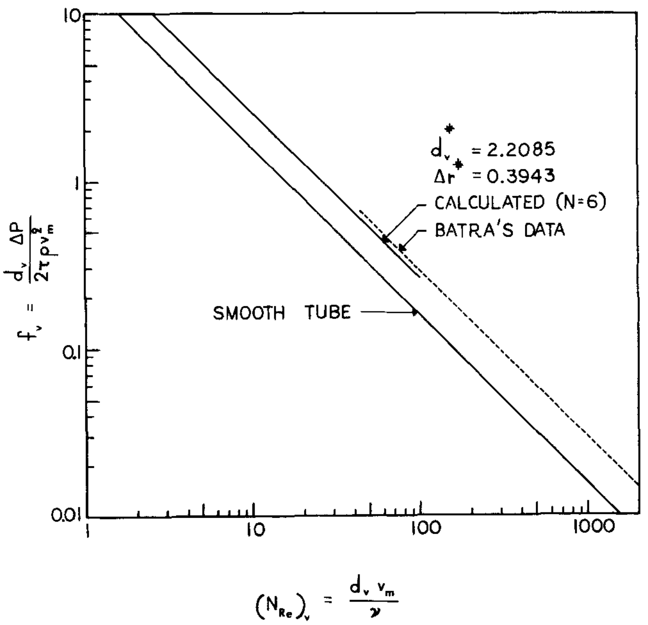

Fig. 11. Comparison of experimental and calculated values of the friction factor of a UPCT.

d_v^* with $\Delta r^* =$ const results in a decrease of f_v, and not an increase. That this is so, can also be seen from limiting situations. As $d_v^* \rightarrow \infty$, the value of f_v has to tend towards the value for a uniform tube, which is the minimum possible value. On the other hand, as d_v^* tends to its minimum possible value, for a given value of Δr^*, $\left[\text{in the present case as } d_v^* \rightarrow \sqrt{\frac{52}{35}}\,\Delta r^*\right]$, the value of the friction factor should increase beyond bound, that is, $f_v \rightarrow \infty$. The source of the aforementioned erroneous conclusion, in addition to experimental errors, seems to be the following. In his experiments Batra used flexible periodically constricted tubes. The ratio τ/d_v was varied by extending or contracting the tubes. Any change in the friction factor was attributed solely to the change in the ratio τ/d_v. This reasoning fails to account for the fact that in the case of a flexible tube a decrease in τ/d_v is accompanied by a simultaneous increase in Δr^*, and, as can be seen in Figure 8, the value of f_v is a strong function of Δr^*, too. Actually, any change in the length of a flexible tube results in changes in both d_v^* and Δr^*, and these changes are such as to produce opposing changes in the values of f_v. Batra's experimental values should be correlated not only with d_v^* alone but with d_v^* and Δr^* simultaneously. At this point, it is perhaps pertinent to state the rather obvious fact that even for fixed values of d_v^* and Δr^* the friction factor of a UPCT is not uniquely determined but that it also depends on the geometry of the wall. There is an infinite variety of wall shapes that have the same d_v^* and Δr^* values.

ACKNOWLEDGMENT

This work was performed under Grant No. 337 8452, New York State Science and Technology Foundation and Grant No. GK-33976, National Science Foundation.

NOTATION

a^* = maximum diameter of the dimensionless periodically constricted tube
d_v = volumetric diameter of a uniform periodically constricted tube
d^* = constriction diameter of the dimensionless periodically constricted tube
d_v^* = dimensionless volumetric diameter of a uniform periodically constricted tube
e_ψ, e_ω = small positive numbers
f_v = friction factor of a uniform periodically constricted tube
f_{U1}^* = profile of ψ^* at a constriction of a uniform periodically constricted tube
f_{U3}^* = profile of ψ^* at a constriction of a uniform periodically constricted tube
f_1^* = profile of ψ^* at the entrance of the extended unit cell
f_2^* = profile of ψ^* at the outlet of the extended unit cell
f_3^* = profile of ω^* at the entrance of the extended unit cell
f_4^* = profile of ω^* at the outlet of the extended unit cell
g = gravitational acceleration
i = index
N = $1/2\kappa^*$ = number of increments per half a unit length
N_{Re} = Reynolds number, defined by Equation (10)
$(N_{Re})_v$ = Reynolds number, defined by Equation (35)
P = pressure, including hydrostatic pressure
P^* = dimensionless pressure
P^+ = normalized pressure
p = absolute pressure
q = flow rate through the periodically constricted tube
r = radial cylindrical coordinate
r^* = dimensionless radial cylindrical coordinate
$r_w^*(z^*)$ = distance of the wall from the axis at z^*
r_1^* = dimensionless radius of the entrance constriction
r_2^* = dimensionless maximum radius
r_3^* = dimensionless radius of the mid constriction
v_m = mean velocity, defined by Equation (34)
v_0 = mean velocity at the entrance constriction
v_r = radial component of the fluid velocity
v_z = axial component of the fluid velocity
$\underline{v}^*$ = dimensionless fluid velocity vector
v_r^* = dimensionless radial component of the fluid velocity
v_z^* = dimensionless axial component of the fluid velocity
w_θ^* = angular component of the dimensionless vorticity
z = axial cylindrical coordinate
z^* = dimensionless axial cylindrical coordinate

Greek Letters

ΔP^* = dimensionless pressure difference at the ends of a segment of a periodically constricted tube
ΔP_1^* = value of ΔP^* for $N_{Re} = 1$
Δr^* = $r_2^* - r_1^*$, amplitude of the projection of the wall on a plane
ζ_ψ, ζ_ω = overrelaxation factors used in the determination of ψ^* and ω^*, respectively
θ = angular cylindrical coordinate
κ^* = increment of the network
ν = kinematic viscosity
ρ = fluid density
ρ_ψ, ρ_ω = weighting factors used in the determination of ψ^* and ω^*, respectively
τ = length of a segment of a periodically constricted tube
ψ = stream function
ψ^* = dimensionless stream function
$\omega^* \equiv E^{*2}\psi^* = r^* w_\theta^*$, arbitrarily defined dependable variable

LITERATURE CITED

Batra, V. K., "Laminar Flow through Wavy Tubes and Wavy Channels," Master's thesis, Univ. Waterloo, Ontario (1969).

Bye, J. A. T., "Obtaining Solutions of the Navier-Stokes Equation by Relaxation Processes," *Comput. J.*, **8**, 53 (1965).

Christiansen, E. B., S. J. Kelsey and T. R. Carter, "Laminar Tube Flow through an Abrupt Contraction," *AIChE J.*, **18**, 372 (1972).

Dullien, F. A. L., and V. K. Batra, "Determination of the Structure of Porous Media," *Ind. Eng. Chem.*, **62**, 25 (1970).

Fromm, J., and F. Harlow, "Numerical Solution of the Problem of Vortex Street Development," *Phys. Fluids*, **6**, 975 (1963).

Greenspan, D., "Numerical Studies of Steady, Viscous, Incompressible Flow in a Channel with a Step," *J. Eng. Math.*, **3**, 21 (1969).

Hamielec, A. E., T. W. Hoffman and L. L. Ross, "Numerical Solution of the Navier-Stokes Equation for Flow Past Spheres: Part I. Viscous Flow around Spheres with and without Radial Mass Efflux," *AIChE J.*, **13**, 212 (1967).

Hamielec, A. E., A. I. Johnson and W. T. Houghton, "Numerical Solution of the Navier-Stokes Equation for Flow Past Spheres: Part II. Viscous Flow around Circulating Spheres at Low Viscosity," *AIChE J.*, **13**, 220 (1967).

Hamielec, A. E., and J. D. Raal, "Numerical Studies of Viscous Flow around Circular Cylinders," *Phys. Fluids*, **12**, 11 (1969).

Houpeurt, A., "Sur l'Écoulement des Gaz dans les Milieux Poreux," (with Eng. transl.) *Rev. Inst. Fr. Petrole Ann. Combust. Liquides*, **14**, 1468 (1959).

Hsu, S.-T., "Turbulent Flow in Wavy Pipes," Ph.D. Dissertation, Univ. Iowa, Iowa City (1968).

Konobeev, V. I., and N. M. Zhavoronkov, "Hydraulic Resistance in Tubes with Wavy Roughness," *Intern. Chem. Eng.*, **2**, 431 (1962).

LeClair, B. P., and A. E. Hamielec, "Viscous Flow through Particle Assemblages at Intermediate Reynolds Numbers. Steady-State Solutions for Flow through Assemblages of Spheres," *Ind. Eng. Chem. Fundamentals*, **7**, 542 (1968).

———., "Viscous Flow through Particle Assemblages at Intermediate Reynolds Numbers. Steady-State Solutions for Flow through Assemblages of Cylinders", *ibid.*, **9**, 608 (1970).

Masliyah, J. H., and N. Epstein, "Steady Symmetric Flow Past Elliptical Cylinders," *ibid.*, **10**, 293 (1971).

Mehta, U. B., and Z. Lavan, "Flow in Two-Dimensional Channel with Rectangular Cavity," NASA C.R.—1245 (1969).

Miller, J. E., et al., "Corrections for Non-uniformity of the Bore of a Capillary Tube Viscometer," Rept. Investig. 6979, Bureau of Mines, U.S. Dept. of the Interior (1967).

Payatakes, A. C., "A New Model for Granular Porous Media. Application to Filtration through Packed Beds", Ph.D. dissertation, Syracuse Univ., New York (1972).

———., Chi Tien and R. M. Turian, "A New Model for Granular Porous Media: Part I. Model Formulation," *AIChE J.*, **18**, 58 (1972).

Petersen, E. E., "Diffusion in a Pore of Varying Cross Section," *ibid.*, **4**, 343 (1958).

Rimon, Y., and S. I. Cheng, "Numerical Solution of a Uniform Flow over a Sphere at Intermediate Reynolds Numbers," *Phys. Fluids*, **12**, 949 (1969).

Runchal, A. K., and M. Wolfshtein, "Numerical Integration Procedure for the Steady State Navier-Stokes Equations," *J. Mech. Eng. Sci.*, **11**, 445 (1969).

Son, J. S., and T. J. Hanratty, "Numerical Solution for the Flow around a Cylinder at Reynolds Numbers of 40, 200 and 500," *J. Fluid Mech.*, **35**, 369 (1969).

Thom, A., Aero. Res. Centre, R. and M., No. 1194 (1929), cited by Masliyah and Epstein (1971).

Vrentas, J. S., J. L. Duda and K. G. Bargeron, "Effect of Axial Diffusion of Vorticity on Flow Development in Circular Conduits: Part I. Numerical Solutions," *AIChE J.*, **12**, 837 (1966).

Wang, Y. L., and P. A. Longwell, "Laminar Flow in the Inlet Section of Parallel Plates," *AIChE J.*, **10**, 323 (1964).

Manuscript received May 16, 1972; revision received August 10, 1972; paper accepted August 11, 1972.

Theoretical Recoveries in Filter Cake Reslurrying and Washing

Theoretical equations and graphs are presented for filter cake reslurrying and washing with linear solute sorption.

In the case of washing, the filter cake was represented by a model consisting of a number of mixing cells in series. Due to the complexity of the model, explicit algebraic solutions could not be obtained for countercurrent washing, but numerical data were generated using a computer.

Practical application of the above is outlined.

A. TOMIAK

The Ontario Paper Company, Ltd.
Thorold, Ontario, Canada

SCOPE

The objective of this work was to provide a unified treatment of filter cake reslurrying and washing theory which would permit a direct comparison of recoveries obtainable in the two cases.

The above is relevant because it provides basic mathematical theory and leads to a series of graphs suitable for routine use in chemical engineering practice.

The theory is based on material balances applied to the case of linear solute distribution between solid and liquid phases. The following systems were considered:

1. Filter cake reslurrying, including simple reslurry shown in Figure 1; multiple reslurries, in which the cake is subjected to successive reslurries with fresh wash liquor; and countercurrent reslurries shown in Figure 2.

2. Filter cake washing, including simple washing, in which the cake is permeated with fresh wash liquor, and countercurrent washing, a particular case of which is shown in Figure 3, in which the cake is subjected to successive washes with the washings obtained in the following stages.

In the selected method of presentation, the emphasis was placed on practical aspects of evaluating the recoveries. As a consequence of this, the theoretical loss equations were represented by a series of graphs included in